06.01.2023[1]

$$\int_0^\infty \frac{dx}{(1+x^\Phi)^\Phi}$$

$\Phi=\frac{1+\sqrt{5}}{2}$ $\quad\Psi=\frac{1-\sqrt{5}}{2}$ $\quad\Phi\Psi=-1$

$x=y^{1-\Phi} \to dx=(1-\Phi)\, y^{-\Phi}dy$

$x^{\Phi} = y^{\Phi(1-\Phi)} = y^{\Phi-\Phi\Phi} = y^{-1} = \frac{1}{y}$ $\qquad 1-\Phi<0 \to$

$$\int_0^\infty \frac{dx}{(1+x^\Phi)^\Phi} = \int_0^\infty (\Phi-1)\frac{y^{-\Phi}}{\left(1+\frac{1}{y}\right)^\Phi}dy = (\Phi-1)\int_0^\infty \frac{1}{(y+1)^\Phi}dy = \frac{\Phi-1}{1-\Phi}(y+1)^{1-\Phi}\Big|_0^\infty = -\frac{\Phi-1}{1-\Phi} = 1$$

oder:

$t=1+x^{\Phi} \leftrightarrow x=(t-1)^{1/\Phi}$

$$\int_0^\infty \frac{dx}{(1+x^\Phi)^\Phi} =$$

$$=\int_1^\infty \frac{1}{t^\Phi}\cdot\frac{1}{\Phi}(t-1)^{\frac{1}{\Phi}-1}dt$$

$$=\frac{1}{\Phi}\int_1^\infty \frac{1}{t^\Phi}\cdot t^{\frac{1}{\Phi}-1}(1-t^{-1})^{\frac{1}{\Phi}-1}dt$$

$$=\frac{1}{\Phi}\int_1^\infty t^{-\Phi+\frac{1}{\Phi}-1}(1-t^{-1})^{\frac{1}{\Phi}-1}dt$$

$t=y^{-1}$

$$=\frac{1}{\Phi}\int_1^0 y^{\Phi-\frac{1}{\Phi}+1}(1-y)^{\frac{1}{\Phi}-1}\cdot(-y^{-2}dy)$$

$$=\frac{1}{\Phi}\int_0^1 y^{\Phi-\frac{1}{\Phi}-1}(1-y)^{\frac{1}{\Phi}-1}dy$$

$$=\frac{1}{\Phi}\, B\left(\Phi-\frac{1}{\Phi},\ \frac{1}{\Phi}\right)$$

$$=\frac{1}{\Phi}\ \frac{\Gamma\left(\Phi-\frac{1}{\Phi}\right)\ \Gamma\left(\frac{1}{\Phi}\right)}{\Gamma(\Phi)}$$

$\frac{1}{\Phi}=\Phi-1 \quad\to$

$$=\frac{1}{\Phi}\cdot\frac{\Gamma(1)\Gamma(\Phi^{-1})}{\Gamma(\Phi^{-1}+1)}$$

$$=\frac{1}{\Phi}\cdot\frac{\Gamma(\Phi^{-1})}{\Phi^{-1}\Gamma(\Phi^{-1})} \quad =1$$

oder:

$x=y^{1/\Phi} = y^{-\Psi}$

$dx=-\Psi\, y^{-\Psi-1}dy \quad\to$

$$\int_0^\infty \frac{dx}{(1+x^\Phi)^\Phi} =$$

$$=-\Psi\int_0^\infty \frac{y^{-\Psi-1}}{(1+y)^\Phi}dy$$

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt \qquad \Re(x)>0,\ \Im(y)>0$$

$x=-\Psi>0,\ x+y=\Phi \to y=\Phi+\Psi=1>0 \to$

$$\int_0^\infty \frac{dx}{(1+x^\Phi)^\Phi} = -\Psi\cdot B(-\Psi,1)$$

$B(-\Psi,1) = -\frac{1}{\Psi}$ $\to$ Integral ist gleich 1

1 https://www.quora.com/How-can-I-evaluate-this-integral-displaystyle-int_-0-infty-frac-dx-left-1-x-phi-right-phi-Here-phi-represents-the-golden-ratio

The so called *Eulerian Integral of the First Kind*:

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$ for real number a, b >0 and the so called *Eulerian Integral of the Second Kind*:

$$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}dx$$ for a real number a > 0.

The integral in (**1**) is known to modern students as the so called *Beta function*, while the integral in (**2**) is known to modern students as the so called *Gamma function*.

Deciding in vacuum which definition of the object at hand is, in a naive way, better, whatever *better* means, or which definition is more natural, or more appropriate or *the right one* is not easy - besides the fact that Euler himself did a fair amount of legwork to identify the shapes of the said integrals in the first place. With the following substitution:

$$x=\frac{y}{1+y}, \quad dx=\frac{dy}{(1+y)^2}$$ the integral (**1**) becomes:

$$B(a,b)=\int_0^\infty \frac{y^{a-1}}{(1+y)^{a+b}}dy$$ and conversely. (4)

With the following substitution:

$$y=\frac{x}{1-x}, \quad dy=\frac{dx}{(1-x)^2}$$ the integral (**4**) becomes:

$$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$$

which is an exact copy of the integral in (**1**). Thus, in the sense described above, the two definitions of the Beta function in (**1**) and in (**4**) are equivalent.

The perceived computational ease or convenience often decides which definition of the Beta function to use.

Lastly, setting b=1 in (**1**), we obtain a run-of-the-mill high school integral:

$$B(a,1)=\int_0^1 x^{a-1}dx = \frac{1}{a}$$

Thus, in his answer Jan simply used $a=-\psi$ which is perfectly fine because $-\psi$ is a strictly positive real number.
Just for kicks, compare the definition of the Gamma function, mostly likely known to you, in (**2**) with the following version used or experimented with by Euler:

$$\Gamma(x)=\frac{1}{x}\prod_{n=1}^{\infty}\frac{\left(1+\frac{x}{n}\right)^x}{1+\frac{x}{n}}$$ for all strictly positive real numbers x > 0.